AQA Maths Further Pure 2

Past Paper Pack

2006-2013

General Certificate of Education
January 2006
Advanced Level Examination

MATHEMATICS **MFP2**
Unit Further Pure 2

Friday 27 January 2006 1.30 pm to 3.00 pm

For this paper you must have:
- an 8-page answer book
- the **blue** AQA booklet of formulae and statistical tables

You may use a graphics calculator.

Time allowed: 1 hour 30 minutes

Instructions
- Use blue or black ink or ball-point pen. Pencil should only be used for drawing.
- Write the information required on the front of your answer book. The *Examining Body* for this paper is AQA. The *Paper Reference* is MFP2.
- Answer **all** questions.
- All necessary working should be shown; otherwise marks for method may be lost.

Information
- The maximum mark for this paper is 75.
- The marks for questions are shown in brackets.

Advice
- Unless stated otherwise, formulae may be quoted, without proof, from the booklet.

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Answer **all** questions.

1 (a) Show that

$$\frac{1}{r^2} - \frac{1}{(r+1)^2} = \frac{2r+1}{r^2(r+1)^2}$$ *(2 marks)*

(b) Hence find the sum of the first n terms of the series

$$\frac{3}{1^2 \times 2^2} + \frac{5}{2^2 \times 3^2} + \frac{7}{3^2 \times 4^2} + \ldots$$ *(4 marks)*

2 The cubic equation

$$x^3 + px^2 + qx + r = 0$$

where p, q and r are real, has roots α, β and γ.

(a) Given that

$$\alpha + \beta + \gamma = 4 \qquad \text{and} \qquad \alpha^2 + \beta^2 + \gamma^2 = 20$$

find the values of p and q. *(5 marks)*

(b) Given further that one root is $3 + \mathrm{i}$, find the value of r. *(5 marks)*

3 The complex numbers z_1 and z_2 are given by

$$z_1 = \frac{1+\mathrm{i}}{1-\mathrm{i}} \qquad \text{and} \qquad z_2 = \frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$$

(a) Show that $z_1 = \mathrm{i}$. *(2 marks)*

(b) Show that $|z_1| = |z_2|$. *(2 marks)*

(c) Express both z_1 and z_2 in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. *(3 marks)*

(d) Draw an Argand diagram to show the points representing z_1, z_2 and $z_1 + z_2$. *(2 marks)*

(e) Use your Argand diagram to show that

$$\tan \frac{5}{12}\pi = 2 + \sqrt{3}$$ *(3 marks)*

4 (a) Prove by induction that

$$2 + (3 \times 2) + (4 \times 2^2) + \ldots + (n+1)\,2^{n-1} = n\,2^n$$

for all integers $n \geqslant 1$. *(6 marks)*

(b) Show that

$$\sum_{r=n+1}^{2n} (r+1)\,2^{r-1} = n\,2^n(2^{n+1} - 1)$$ *(3 marks)*

5 The complex number z satisfies the relation

$$|z + 4 - 4\mathrm{i}| = 4$$

(a) Sketch, on an Argand diagram, the locus of z. *(3 marks)*

(b) Show that the greatest value of $|z|$ is $4\,(\sqrt{2} + 1)$. *(3 marks)*

(c) Find the value of z for which

$$\arg(z + 4 - 4\mathrm{i}) = \tfrac{1}{6}\pi$$

Give your answer in the form $a + \mathrm{i}b$. *(3 marks)*

Turn over for the next question

6 It is given that $z = \mathrm{e}^{\mathrm{i}\theta}$.

(a) (i) Show that

$$z + \frac{1}{z} = 2\cos\theta$$ *(2 marks)*

(ii) Find a similar expression for

$$z^2 + \frac{1}{z^2}$$ *(2 marks)*

(iii) Hence show that

$$z^2 - z + 2 - \frac{1}{z} + \frac{1}{z^2} = 4\cos^2\theta - 2\cos\theta$$ *(3 marks)*

(b) Hence solve the quartic equation

$$z^4 - z^3 + 2z^2 - z + 1 = 0$$

giving the roots in the form $a + \mathrm{i}b$. *(5 marks)*

7 (a) Use the definitions

$$\sinh\theta = \frac{1}{2}(\mathrm{e}^{\theta} - \mathrm{e}^{-\theta}) \qquad \text{and} \qquad \cosh\theta = \frac{1}{2}(\mathrm{e}^{\theta} + \mathrm{e}^{-\theta})$$

to show that:

(i) $2\sinh\theta\cosh\theta = \sinh 2\theta$; *(2 marks)*

(ii) $\cosh^2\theta + \sinh^2\theta = \cosh 2\theta$. *(3 marks)*

(b) A curve is given parametrically by

$$x = \cosh^3\theta, \quad y = \sinh^3\theta$$

(i) Show that

$$\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2 = \frac{9}{4}\sinh^2 2\theta\cosh 2\theta$$ *(6 marks)*

(ii) Show that the length of the arc of the curve from the point where $\theta = 0$ to the point where $\theta = 1$ is

$$\frac{1}{2}\left[(\cosh 2)^{\frac{3}{2}} - 1\right]$$ *(6 marks)*

END OF QUESTIONS

General Certificate of Education
June 2006
Advanced Level Examination

MATHEMATICS **MFP2**
Unit Further Pure 2

Monday 19 June 2006 9.00 am to 10.30 am

For this paper you must have:
- an 8-page answer book
- the **blue** AQA booklet of formulae and statistical tables

You may use a graphics calculator.

Time allowed: 1 hour 30 minutes

Instructions
- Use blue or black ink or ball-point pen. Pencil should only be used for drawing.
- Write the information required on the front of your answer book. The *Examining Body* for this paper is AQA. The *Paper Reference* is MFP2.
- Answer **all** questions.
- Show all necessary working; otherwise marks for method may be lost.

Information
- The maximum mark for this paper is 75.
- The marks for questions are shown in brackets.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

Answer **all** questions.

1 (a) Given that

$$\frac{r^2 + r - 1}{r(r+1)} = A + B\left(\frac{1}{r} - \frac{1}{r+1}\right)$$

find the values of A and B. *(3 marks)*

(b) Hence find the value of

$$\sum_{r=1}^{99} \frac{r^2 + r - 1}{r(r+1)}$$

(4 marks)

2 A curve has parametric equations

$$x = t - \frac{1}{3}t^3, \quad y = t^2$$

(a) Show that

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = (1 + t^2)^2$$

(3 marks)

(b) The arc of the curve between $t = 1$ and $t = 2$ is rotated through 2π radians about the x-axis.

Show that S, the surface area generated, is given by $S = k\pi$, where k is a rational number to be found. *(5 marks)*

3 The curve C has equation

$$y = \cosh x - 3\sinh x$$

(a) (i) The line $y = -1$ meets C at the point $(k, -1)$.

Show that

$$e^{2k} - e^{k} - 2 = 0$$ *(3 marks)*

(ii) Hence find k, giving your answer in the form $\ln a$. *(4 marks)*

(b) (i) Find the x-coordinate of the point where the curve C intersects the x-axis, giving your answer in the form $p \ln a$. *(4 marks)*

(ii) Show that C has no stationary points. *(3 marks)*

(iii) Show that there is exactly one point on C for which $\dfrac{d^2y}{dx^2} = 0$. *(1 mark)*

4 (a) On one Argand diagram, sketch the locus of points satisfying:

(i) $|z - 3 + 2i| = 4$; *(3 marks)*

(ii) $\arg(z - 1) = -\frac{1}{4}\pi$. *(3 marks)*

(b) Indicate on your sketch the set of points satisfying both

$$|z - 3 + 2i| \leqslant 4$$

and $$\arg(z - 1) = -\frac{1}{4}\pi$$ *(1 mark)*

Turn over for the next question

5 The cubic equation

$$z^3 - 4iz^2 + qz - (4 - 2i) = 0$$

where q is a complex number, has roots α, β and γ.

(a) Write down the value of:

(i) $\alpha + \beta + \gamma$; *(1 mark)*

(ii) $\alpha\beta\gamma$. *(1 mark)*

(b) Given that $\alpha = \beta + \gamma$, show that:

(i) $\alpha = 2i$; *(1 mark)*

(ii) $\beta\gamma = -(1 + 2i)$; *(2 marks)*

(iii) $q = -(5 + 2i)$. *(3 marks)*

(c) Show that β and γ are the roots of the equation

$$z^2 - 2iz - (1 + 2i) = 0$$ *(2 marks)*

(d) Given that β is real, find β and γ. *(3 marks)*

6 (a) The function f is given by

$$f(n) = 15^n - 8^{n-2}$$

Express

$$f(n+1) - 8f(n)$$

in the form $k \times 15^n$. *(4 marks)*

(b) Prove by induction that $15^n - 8^{n-2}$ is a multiple of 7 for all integers $n \geqslant 2$. *(4 marks)*

7 (a) Find the six roots of the equation $z^6 = 1$, giving your answers in the form $e^{i\phi}$, where $-\pi < \phi \leqslant \pi$. *(3 marks)*

(b) It is given that $w = e^{i\theta}$, where $\theta \neq n\pi$.

(i) Show that $\dfrac{w^2 - 1}{w} = 2i \sin \theta$. *(2 marks)*

(ii) Show that $\dfrac{w}{w^2 - 1} = -\dfrac{i}{2 \sin \theta}$. *(2 marks)*

(iii) Show that $\dfrac{2i}{w^2 - 1} = \cot \theta - i$. *(3 marks)*

(iv) Given that $z = \cot \theta - i$, show that $z + 2i = zw^2$. *(2 marks)*

(c) (i) Explain why the equation

$$(z + 2i)^6 = z^6$$

has five roots. *(1 mark)*

(ii) Find the five roots of the equation

$$(z + 2i)^6 = z^6$$

giving your answers in the form $a + ib$. *(4 marks)*

END OF QUESTIONS

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General Certificate of Education
January 2007
Advanced Level Examination

MATHEMATICS **MFP2**
Unit Further Pure 2

Thursday 1 February 2007 9.00 am to 10.30 am

For this paper you must have:
- an 8-page answer book
- the **blue** AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed: 1 hour 30 minutes

Instructions
- Use blue or black ink or ball-point pen. Pencil should only be used for drawing.
- Write the information required on the front of your answer book. The *Examining Body* for this paper is AQA. The *Paper Reference* is MFP2.
- Answer **all** questions.
- Show all necessary working; otherwise marks for method may be lost.

Information
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Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

Answer **all** questions.

1 (a) Given that

$$4\cosh^2 x = 7\sinh x + 1$$

find the two possible values of $\sinh x$. *(4 marks)*

(b) Hence obtain the two possible values of x, giving your answers in the form $\ln p$. *(3 marks)*

2 (a) Sketch on one diagram:

(i) the locus of points satisfying $|z - 4 + 2\mathrm{i}| = 2$; *(3 marks)*

(ii) the locus of points satisfying $|z| = |z - 3 - 2\mathrm{i}|$. *(3 marks)*

(b) Shade on your sketch the region in which

both $|z - 4 + 2\mathrm{i}| \leqslant 2$

and $|z| \leqslant |z - 3 - 2\mathrm{i}|$ *(2 marks)*

3 The cubic equation

$$z^3 + 2(1 - \mathrm{i})z^2 + 32(1 + \mathrm{i}) = 0$$

has roots α, β and γ.

(a) It is given that α is of the form $k\mathrm{i}$, where k is real. By substituting $z = k\mathrm{i}$ into the equation, show that $k = 4$. *(5 marks)*

(b) Given that $\beta = -4$, find the value of γ. *(2 marks)*

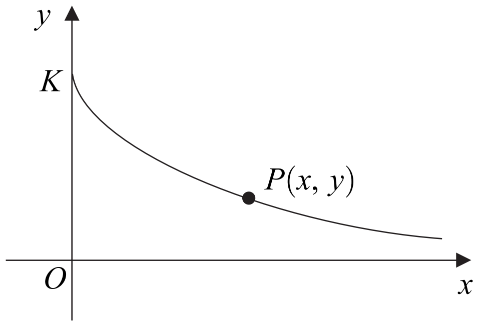
4 (a) Given that $y = \operatorname{sech} t$, show that:

(i) $\dfrac{\mathrm{d}y}{\mathrm{d}t} = -\operatorname{sech} t \tanh t\,;$ *(3 marks)*

(ii) $\left(\dfrac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = \operatorname{sech}^2 t - \operatorname{sech}^4 t\,.$ *(2 marks)*

(b) The diagram shows a sketch of part of the curve given parametrically by

$$x = t - \tanh t \qquad y = \operatorname{sech} t$$



The curve meets the y-axis at the point K, and $P(x, y)$ is a general point on the curve. The arc length KP is denoted by s. Show that:

(i) $\left(\dfrac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\dfrac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = \tanh^2 t\,;$ *(4 marks)*

(ii) $s = \ln \cosh t\,;$ *(3 marks)*

(iii) $y = \mathrm{e}^{-s}\,.$ *(2 marks)*

(c) The arc KP is rotated through 2π radians about the x-axis. Show that the surface area generated is

$$2\pi(1 - \mathrm{e}^{-s})$$ *(4 marks)*

Turn over for the next question

5 (a) Prove by induction that, if n is a positive integer,

$$(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta$$ *(5 marks)*

(b) Find the value of $\left(\cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6}\right)^6$. *(2 marks)*

(c) Show that

$$(\cos\theta + \mathrm{i}\sin\theta)(1 + \cos\theta - \mathrm{i}\sin\theta) = 1 + \cos\theta + \mathrm{i}\sin\theta$$ *(3 marks)*

(d) Hence show that

$$\left(1 + \cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6}\right)^6 + \left(1 + \cos\frac{\pi}{6} - \mathrm{i}\sin\frac{\pi}{6}\right)^6 = 0$$ *(4 marks)*

6 (a) Find the three roots of $z^3 = 1$, giving the non-real roots in the form $\mathrm{e}^{\mathrm{i}\theta}$, where $-\pi < \theta \leqslant \pi$. *(2 marks)*

(b) Given that ω is one of the non-real roots of $z^3 = 1$, show that

$$1 + \omega + \omega^2 = 0$$ *(2 marks)*

(c) By using the result in part (b), or otherwise, show that:

(i) $\dfrac{\omega}{\omega + 1} = -\dfrac{1}{\omega}$; *(2 marks)*

(ii) $\dfrac{\omega^2}{\omega^2 + 1} = -\omega$; *(1 mark)*

(iii) $\left(\dfrac{\omega}{\omega + 1}\right)^k + \left(\dfrac{\omega^2}{\omega^2 + 1}\right)^k = (-1)^k\, 2\cos\frac{2}{3}k\pi$, where k is an integer. *(5 marks)*

7 (a) Use the identity $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$ with $A = (r+1)x$ and $B = rx$ to show that

$$\tan rx \tan(r+1)x = \frac{\tan(r+1)x}{\tan x} - \frac{\tan rx}{\tan x} - 1$$ *(4 marks)*

(b) Use the method of differences to show that

$$\tan\frac{\pi}{50}\tan\frac{2\pi}{50} + \tan\frac{2\pi}{50}\tan\frac{3\pi}{50} + \ldots + \tan\frac{19\pi}{50}\tan\frac{20\pi}{50} = \frac{\tan\dfrac{2\pi}{5}}{\tan\dfrac{\pi}{50}} - 20$$ *(5 marks)*

END OF QUESTIONS

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General Certificate of Education
June 2007
Advanced Level Examination

MATHEMATICS **MFP2**
Unit Further Pure 2

Tuesday 26 June 2007 1.30 pm to 3.00 pm

For this paper you must have:
- an 8-page answer book
- the **blue** AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed: 1 hour 30 minutes

Instructions
- Use blue or black ink or ball-point pen. Pencil should only be used for drawing.
- Write the information required on the front of your answer book. The *Examining Body* for this paper is AQA. The *Paper Reference* is MFP2.
- Answer **all** questions.
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Information
- The maximum mark for this paper is 75.
- The marks for questions are shown in brackets.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

Answer **all** questions.

1 (a) Given that $f(r) = (r-1)r^2$, show that

$$f(r+1) - f(r) = r(3r+1)$$ *(3 marks)*

(b) Use the method of differences to find the value of

$$\sum_{r=50}^{99} r(3r+1)$$ *(4 marks)*

2 The cubic equation

$$z^3 + pz^2 + 6z + q = 0$$

has roots α, β and γ.

(a) Write down the value of $\alpha\beta + \beta\gamma + \gamma\alpha$. *(1 mark)*

(b) Given that p and q are real and that $\alpha^2 + \beta^2 + \gamma^2 = -12$:

(i) explain why the cubic equation has two non-real roots and one real root; *(2 marks)*

(ii) find the value of p. *(4 marks)*

(c) One root of the cubic equation is $-1 + 3\mathrm{i}$.

Find:

(i) the other two roots; *(3 marks)*

(ii) the value of q. *(2 marks)*

3 Use De Moivre's Theorem to find the smallest positive angle θ for which

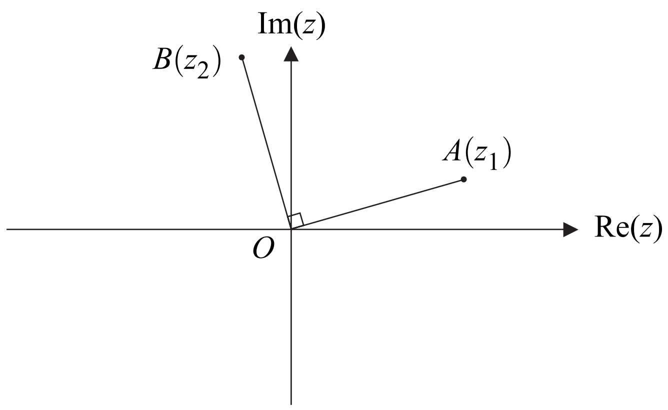
$$(\cos\theta + \mathrm{i}\sin\theta)^{15} = -\mathrm{i}$$ *(5 marks)*

4 (a) Differentiate $x\tan^{-1}x$ with respect to x. *(2 marks)*

(b) Show that

$$\int_0^1 \tan^{-1}x \, \mathrm{d}x = \frac{\pi}{4} - \ln\sqrt{2}$$ *(5 marks)*

5 The sketch shows an Argand diagram. The points A and B represent the complex numbers z_1 and z_2 respectively. The angle $AOB = 90°$ and $OA = OB$.



(a) Explain why $z_2 = \mathrm{i}z_1$. *(2 marks)*

(b) On a **single** copy of the diagram, draw:

(i) the locus L_1 of points satisfying $|z - z_2| = |z - z_1|$; *(2 marks)*

(ii) the locus L_2 of points satisfying $\arg(z - z_2) = \arg z_1$. *(3 marks)*

(c) Find, in terms of z_1, the complex number representing the point of intersection of L_1 and L_2. *(2 marks)*

6 (a) Show that

$$\left(1 - \frac{1}{(k+1)^2}\right) \times \frac{k+1}{2k} = \frac{k+2}{2(k+1)}$$ *(3 marks)*

(b) Prove by induction that for all integers $n \geqslant 2$

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\ldots\left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}$$ *(4 marks)*

Turn over for the next question

7 A curve has equation $y = 4\sqrt{x}$.

(a) Show that the length of arc s of the curve between the points where $x = 0$ and $x = 1$ is given by

$$s = \int_0^1 \sqrt{\frac{x+4}{x}}\, \mathrm{d}x$$ *(4 marks)*

(b) (i) Use the substitution $x = 4\sinh^2\theta$ to show that

$$\int \sqrt{\frac{x+4}{x}}\, \mathrm{d}x = \int 8\cosh^2\theta\, \mathrm{d}\theta$$ *(5 marks)*

(ii) Hence show that

$$s = 4\sinh^{-1} 0.5 + \sqrt{5}$$ *(6 marks)*

8 (a) (i) Given that $z^6 - 4z^3 + 8 = 0$, show that $z^3 = 2 \pm 2\mathrm{i}$. *(2 marks)*

(ii) Hence solve the equation

$$z^6 - 4z^3 + 8 = 0$$

giving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. *(6 marks)*

(b) Show that, for any real values of k and θ,

$$(z - k\mathrm{e}^{\mathrm{i}\theta})(z - k\mathrm{e}^{-\mathrm{i}\theta}) = z^2 - 2kz\cos\theta + k^2$$ *(2 marks)*

(c) Express $z^6 - 4z^3 + 8$ as the product of three quadratic factors with real coefficients. *(3 marks)*

END OF QUESTIONS

General Certificate of Education
January 2008
Advanced Level Examination

MATHEMATICS **MFP2**
Unit Further Pure 2

Thursday 31 January 2008 9.00 am to 10.30 am

For this paper you must have:
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Time allowed: 1 hour 30 minutes

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Information
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Advice
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Answer **all** questions.

1 (a) Express $4 + 4\mathrm{i}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. *(3 marks)*

(b) Solve the equation

$$z^5 = 4 + 4\mathrm{i}$$

giving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. *(5 marks)*

2 (a) Show that

$$(2r+1)^3 - (2r-1)^3 = 24r^2 + 2$$ *(3 marks)*

(b) Hence, using the method of differences, show that

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$ *(6 marks)*

3 A circle C and a half-line L have equations

$$|z - 2\sqrt{3} - \mathrm{i}| = 4$$

and

$$\arg(z + \mathrm{i}) = \frac{\pi}{6}$$

respectively.

(a) Show that:

(i) the circle C passes through the point where $z = -\mathrm{i}$; *(2 marks)*

(ii) the half-line L passes through the centre of C. *(3 marks)*

(b) On one Argand diagram, sketch C and L. *(4 marks)*

(c) Shade on your sketch the set of points satisfying both

$$|z - 2\sqrt{3} - \mathrm{i}| \leqslant 4$$

and

$$0 \leqslant \arg(z + \mathrm{i}) \leqslant \frac{\pi}{6}$$ *(2 marks)*

4 The cubic equation

$$z^3 + \mathrm{i}z^2 + 3z - (1 + \mathrm{i}) = 0$$

has roots α, β and γ.

(a) Write down the value of:

(i) $\alpha + \beta + \gamma$; *(1 mark)*

(ii) $\alpha\beta + \beta\gamma + \gamma\alpha$; *(1 mark)*

(iii) $\alpha\beta\gamma$. *(1 mark)*

(b) Find the value of:

(i) $\alpha^2 + \beta^2 + \gamma^2$; *(3 marks)*

(ii) $\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2$; *(4 marks)*

(iii) $\alpha^2\beta^2\gamma^2$. *(2 marks)*

(c) Hence write down a cubic equation whose roots are α^2, β^2 and γ^2. *(2 marks)*

5 Prove by induction that for all integers $n \geqslant 1$

$$\sum_{r=1}^{n} (r^2 + 1)(r!) = n(n + 1)!$$ *(7 marks)*

Turn over for the next question

6 (a) (i) By applying De Moivre's theorem to $(\cos\theta + \mathrm{i}\sin\theta)^3$, show that

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$ *(3 marks)*

(ii) Find a similar expression for $\sin 3\theta$. *(1 mark)*

(iii) Deduce that

$$\tan 3\theta = \frac{\tan^3\theta - 3\tan\theta}{3\tan^2\theta - 1}$$ *(3 marks)*

(b) (i) Hence show that $\tan\dfrac{\pi}{12}$ is a root of the cubic equation

$$x^3 - 3x^2 - 3x + 1 = 0$$ *(3 marks)*

(ii) Find two other values of θ, where $0 < \theta < \pi$, for which $\tan\theta$ is a root of this cubic equation. *(2 marks)*

(c) Hence show that

$$\tan\frac{\pi}{12} + \tan\frac{5\pi}{12} = 4$$ *(2 marks)*

7 (a) Given that $y = \ln\tanh\dfrac{x}{2}$, where $x > 0$, show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \operatorname{cosech} x$$ *(6 marks)*

(b) A curve has equation $y = \ln\tanh\dfrac{x}{2}$, where $x > 0$. The length of the arc of the curve between the points where $x = 1$ and $x = 2$ is denoted by s.

(i) Show that

$$s = \int_1^2 \coth x\,\mathrm{d}x$$ *(2 marks)*

(ii) Hence show that $s = \ln(2\cosh 1)$. *(4 marks)*

END OF QUESTIONS

General Certificate of Education
June 2008
Advanced Level Examination

MATHEMATICS **MFP2**
Unit Further Pure 2

Thursday 15 May 2008 9.00 am to 10.30 am

For this paper you must have:
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You may use a graphics calculator.

Time allowed: 1 hour 30 minutes

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Information
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Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

Answer **all** questions.

1 (a) Express

$$5\sinh x + \cosh x$$

in the form $Ae^{x} + Be^{-x}$, where A and B are integers. *(2 marks)*

(b) Solve the equation

$$5\sinh x + \cosh x + 5 = 0$$

giving your answer in the form $\ln a$, where a is a rational number. *(4 marks)*

2 (a) Given that

$$\frac{1}{r(r+1)(r+2)} = \frac{A}{r(r+1)} + \frac{B}{(r+1)(r+2)}$$

show that $A = \frac{1}{2}$ and find the value of B. *(3 marks)*

(b) Use the method of differences to find

$$\sum_{r=10}^{98} \frac{1}{r(r+1)(r+2)}$$

giving your answer as a rational number. *(4 marks)*

3 The cubic equation

$$z^3 + qz + (18 - 12\mathrm{i}) = 0$$

where q is a complex number, has roots α, β and γ.

(a) Write down the value of:

(i) $\alpha\beta\gamma$; *(1 mark)*

(ii) $\alpha + \beta + \gamma$. *(1 mark)*

(b) Given that $\beta + \gamma = 2$, find the value of:

(i) α; *(1 mark)*

(ii) $\beta\gamma$; *(2 marks)*

(iii) q. *(3 marks)*

(c) Given that β is of the form $k\mathrm{i}$, where k is real, find β and γ. *(4 marks)*

4 (a) A circle C in the Argand diagram has equation

$$|z + 5 - \mathrm{i}| = \sqrt{2}$$

Write down its radius and the complex number representing its centre. *(2 marks)*

(b) A half-line L in the Argand diagram has equation

$$\arg(z + 2\mathrm{i}) = \frac{3\pi}{4}$$

Show that $z_1 = -4 + 2\mathrm{i}$ lies on L. *(2 marks)*

(c) (i) Show that $z_1 = -4 + 2\mathrm{i}$ also lies on C. *(1 mark)*

(ii) Hence show that L touches C. *(3 marks)*

(iii) Sketch L and C on one Argand diagram. *(2 marks)*

(d) The complex number z_2 lies on C and is such that $\arg(z_2 + 2\mathrm{i})$ has as great a value as possible.

Indicate the position of z_2 on your sketch. *(2 marks)*

Turn over ▶

5 (a) Use the definition $\cosh x = \frac{1}{2}(e^x + e^{-x})$ to show that $\cosh 2x = 2\cosh^2 x - 1$. *(2 marks)*

(b) (i) The arc of the curve $y = \cosh x$ between $x = 0$ and $x = \ln a$ is rotated through 2π radians about the x-axis. Show that S, the surface area generated, is given by

$$S = 2\pi \int_0^{\ln a} \cosh^2 x \, dx$$ *(3 marks)*

(ii) Hence show that

$$S = \pi\left(\ln a + \frac{a^4 - 1}{4a^2}\right)$$ *(5 marks)*

6 By using the substitution $u = x - 2$, or otherwise, find the exact value of

$$\int_{-1}^{5} \frac{dx}{\sqrt{32 + 4x - x^2}}$$ *(5 marks)*

7 (a) Explain why $n(n+1)$ is a multiple of 2 when n is an integer. *(1 mark)*

(b) (i) Given that

$$f(n) = n(n^2 + 5)$$

show that $f(k+1) - f(k)$, where k is a positive integer, is a multiple of 6. *(4 marks)*

(ii) Prove by induction that $f(n)$ is a multiple of 6 for all integers $n \geqslant 1$. *(4 marks)*

8 (a) (i) Expand

$$\left(z+\frac{1}{z}\right)\left(z-\frac{1}{z}\right)$$ *(1 mark)*

(ii) Hence, or otherwise, expand

$$\left(z+\frac{1}{z}\right)^4\left(z-\frac{1}{z}\right)^2$$ *(3 marks)*

(b) (i) Use De Moivre's theorem to show that if $z = \cos\theta + \mathrm{i}\sin\theta$ then

$$z^n + \frac{1}{z^n} = 2\cos n\theta$$ *(3 marks)*

(ii) Write down a corresponding result for $z^n - \dfrac{1}{z^n}$. *(1 mark)*

(c) Hence express $\cos^4\theta\sin^2\theta$ in the form

$$A\cos 6\theta + B\cos 4\theta + C\cos 2\theta + D$$

where A, B, C and D are rational numbers. *(4 marks)*

(d) Find $\displaystyle\int \cos^4\theta\sin^2\theta\ \mathrm{d}\theta$. *(2 marks)*

END OF QUESTIONS

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General Certificate of Education
January 2009
Advanced Level Examination

MATHEMATICS **MFP2**
Unit Further Pure 2

Monday 19 January 2009 1.30 pm to 3.00 pm

For this paper you must have:
- an 8-page answer book
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed: 1 hour 30 minutes

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Write the information required on the front of your answer book. The *Examining Body* for this paper is AQA. The *Paper Reference* is MFP2.
- Answer **all** questions.
- Show all necessary working; otherwise marks for method may be lost.

Information
- The maximum mark for this paper is 75.
- The marks for questions are shown in brackets.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

Answer **all** questions.

1 (a) Use the definitions $\sinh\theta = \frac{1}{2}(e^{\theta} - e^{-\theta})$ and $\cosh\theta = \frac{1}{2}(e^{\theta} + e^{-\theta})$ to show that

$$1 + 2\sinh^2\theta = \cosh 2\theta$$ *(3 marks)*

(b) Solve the equation

$$3\cosh 2\theta = 2\sinh\theta + 11$$

giving each of your answers in the form $\ln p$. *(6 marks)*

2 (a) Indicate on an Argand diagram the region for which $|z - 4\mathrm{i}| \leqslant 2$. *(4 marks)*

(b) The complex number z satisfies $|z - 4\mathrm{i}| \leqslant 2$. Find the range of possible values of $\arg z$. *(4 marks)*

3 (a) Given that $\mathrm{f}(r) = \frac{1}{4}r^2(r+1)^2$, show that

$$\mathrm{f}(r) - \mathrm{f}(r-1) = r^3$$ *(3 marks)*

(b) Use the method of differences to show that

$$\sum_{r=n}^{2n} r^3 = \frac{3}{4}n^2(n+1)(5n+1)$$ *(5 marks)*

4 It is given that α, β and γ satisfy the equations

$$\begin{aligned} \alpha + \beta + \gamma &= 1 \\ \alpha^2 + \beta^2 + \gamma^2 &= -5 \\ \alpha^3 + \beta^3 + \gamma^3 &= -23 \end{aligned}$$

(a) Show that $\alpha\beta + \beta\gamma + \gamma\alpha = 3$. *(3 marks)*

(b) Use the identity

$$(\alpha + \beta + \gamma)(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha) = \alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma$$

to find the value of $\alpha\beta\gamma$. *(2 marks)*

(c) Write down a cubic equation, with integer coefficients, whose roots are α, β and γ. *(2 marks)*

(d) Explain why this cubic equation has two non-real roots. *(2 marks)*

(e) Given that α is real, find the values of α, β and γ. *(4 marks)*

5 (a) Given that $u = \cosh^2 x$, show that $\dfrac{\mathrm{d}u}{\mathrm{d}x} = \sinh 2x$. *(2 marks)*

(b) Hence show that

$$\int_0^1 \frac{\sinh 2x}{1 + \cosh^4 x}\,\mathrm{d}x = \tan^{-1}(\cosh^2 1) - \frac{\pi}{4}$$ *(5 marks)*

6 Prove by induction that

$$\frac{2 \times 1}{2 \times 3} + \frac{2^2 \times 2}{3 \times 4} + \frac{2^3 \times 3}{4 \times 5} + \ldots + \frac{2^n \times n}{(n+1)(n+2)} = \frac{2^{n+1}}{n+2} - 1$$

for all integers $n \geqslant 1$. *(7 marks)*

Turn over for the next question

7 (a) Show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\cosh^{-1}\frac{1}{x}\right) = \frac{-1}{x\sqrt{1-x^2}}$$ *(3 marks)*

(b) A curve has equation

$$y = \sqrt{1-x^2} - \cosh^{-1}\frac{1}{x} \qquad (0 < x < 1)$$

Show that:

(i) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\sqrt{1-x^2}}{x}$; *(4 marks)*

(ii) the length of the arc of the curve from the point where $x = \frac{1}{4}$ to the point where $x = \frac{3}{4}$ is $\ln 3$. *(5 marks)*

8 (a) Show that

$$(z^4 - \mathrm{e}^{\mathrm{i}\theta})(z^4 - \mathrm{e}^{-\mathrm{i}\theta}) = z^8 - 2z^4\cos\theta + 1$$ *(2 marks)*

(b) Hence solve the equation

$$z^8 - z^4 + 1 = 0$$

giving your answers in the form $\mathrm{e}^{\mathrm{i}\phi}$, where $-\pi < \phi \leqslant \pi$. *(6 marks)*

(c) Indicate the roots on an Argand diagram. *(3 marks)*

END OF QUESTIONS

General Certificate of Education
June 2009
Advanced Level Examination

MATHEMATICS **MFP2**
Unit Further Pure 2

Friday 5 June 2009 1.30 pm to 3.00 pm

For this paper you must have:
- an 8-page answer book
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed: 1 hour 30 minutes

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Write the information required on the front of your answer book. The *Examining Body* for this paper is AQA. The *Paper Reference* is MFP2.
- Answer **all** questions.
- Show all necessary working; otherwise marks for method may be lost.

Information
- The maximum mark for this paper is 75.
- The marks for questions are shown in brackets.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

Answer **all** questions.

1 Given that $z = 2\text{e}^{\frac{\pi \text{i}}{12}}$ satisfies the equation

$$z^4 = a(1 + \sqrt{3}\,\text{i})$$

where a is real:

(a) find the value of a; *(3 marks)*

(b) find the other three roots of this equation, giving your answers in the form $r\text{e}^{\text{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. *(5 marks)*

2 (a) Given that

$$\frac{1}{4r^2 - 1} = \frac{A}{2r - 1} + \frac{B}{2r + 1}$$

find the values of A and B. *(2 marks)*

(b) Use the method of differences to show that

$$\sum_{r=1}^{n} \frac{1}{4r^2 - 1} = \frac{n}{2n + 1}$$ *(3 marks)*

(c) Find the least value of n for which $\sum_{r=1}^{n} \frac{1}{4r^2 - 1}$ differs from 0.5 by less than 0.001. *(3 marks)*

3 The cubic equation

$$z^3 + pz^2 + 25z + q = 0$$

where p and q are real, has a root $\alpha = 2 - 3\text{i}$.

(a) Write down another non-real root, β, of this equation. *(1 mark)*

(b) Find:

(i) the value of $\alpha\beta$; *(1 mark)*

(ii) the third root, γ, of the equation; *(3 marks)*

(iii) the values of p and q. *(3 marks)*

4 (a) Sketch the graph of $y = \tanh x$. *(2 marks)*

(b) Given that $u = \tanh x$, use the definitions of $\sinh x$ and $\cosh x$ in terms of e^x and e^{-x} to show that

$$x = \frac{1}{2}\ln\left(\frac{1+u}{1-u}\right)$$ *(6 marks)*

(c) (i) Show that the equation

$$3\operatorname{sech}^2 x + 7\tanh x = 5$$

can be written as

$$3\tanh^2 x - 7\tanh x + 2 = 0$$ *(2 marks)*

(ii) Show that the equation

$$3\tanh^2 x - 7\tanh x + 2 = 0$$

has only one solution for x.

Find this solution in the form $\frac{1}{2}\ln a$, where a is an integer. *(5 marks)*

5 (a) Prove by induction that, if n is a positive integer,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$ *(5 marks)*

(b) Hence, given that

$$z = \cos\theta + i\sin\theta$$

show that

$$z^n + \frac{1}{z^n} = 2\cos n\theta$$ *(3 marks)*

(c) Given further that $z + \frac{1}{z} = \sqrt{2}$, find the value of

$$z^{10} + \frac{1}{z^{10}}$$ *(4 marks)*

Turn over for the next question

6 (a) Two points, A and B, on an Argand diagram are represented by the complex numbers $2 + 3\mathrm{i}$ and $-4 - 5\mathrm{i}$ respectively. Given that the points A and B are at the ends of a diameter of a circle C_1, express the equation of C_1 in the form $|z - z_0| = k$. *(4 marks)*

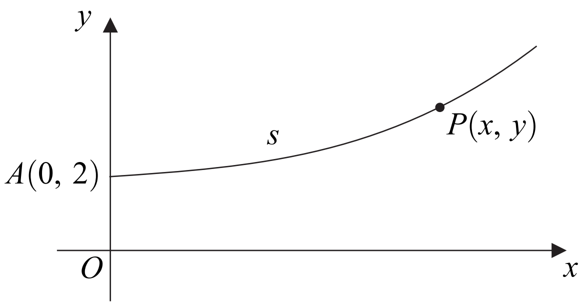
(b) A second circle, C_2, is represented on the Argand diagram by the equation $|z - 5 + 4\mathrm{i}| = 4$. Sketch on one Argand diagram both C_1 and C_2. *(3 marks)*

(c) The points representing the complex numbers z_1 and z_2 lie on C_1 and C_2 respectively and are such that $|z_1 - z_2|$ has its maximum value. Find this maximum value, giving your answer in the form $a + b\sqrt{5}$. *(5 marks)*

7 The diagram shows a curve which starts from the point A with coordinates (0, 2). The curve is such that, at every point P on the curve,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}s$$

where s is the length of the arc AP.



(a) (i) Show that

$$\frac{\mathrm{d}s}{\mathrm{d}x} = \frac{1}{2}\sqrt{4 + s^2}$$ *(3 marks)*

(ii) Hence show that

$$s = 2\sinh\frac{x}{2}$$ *(4 marks)*

(iii) Hence find the cartesian equation of the curve. *(3 marks)*

(b) Show that

$$y^2 = 4 + s^2$$ *(2 marks)*

END OF QUESTIONS

General Certificate of Education
Advanced Level Examination
January 2010

Mathematics MFP2

Unit Further Pure 2

Friday 15 January 2010 1.30 pm to 3.00 pm

For this paper you must have:
- an 8-page answer book
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed
- 1 hour 30 minutes

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Write the information required on the front of your answer book. The **Examining Body** for this paper is AQA. The **Paper Reference** is MFP2.
- Answer **all** questions.
- Show all necessary working; otherwise marks for method may be lost.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 75.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

Answer **all** questions.

1 (a) Use the definitions $\cosh x = \frac{1}{2}(e^x + e^{-x})$ and $\sinh x = \frac{1}{2}(e^x - e^{-x})$ to show that

$$\cosh^2 x - \sinh^2 x = 1$$

(3 marks)

(b) (i) Express

$$5\cosh^2 x + 3\sinh^2 x$$

in terms of $\cosh x$. *(1 mark)*

(ii) Sketch the curve $y = \cosh x$. *(1 mark)*

(iii) Hence solve the equation

$$5\cosh^2 x + 3\sinh^2 x = 9.5$$

giving your answers in logarithmic form. *(4 marks)*

2 (a) On the same Argand diagram, draw:

(i) the locus of points satisfying $|z - 4 + 2i| = 4$; *(3 marks)*

(ii) the locus of points satisfying $|z| = |z - 2i|$. *(3 marks)*

(b) Indicate on your sketch the set of points satisfying both

$$|z - 4 + 2i| \leqslant 4$$

and

$$|z| \geqslant |z - 2i|$$

(2 marks)

3 The cubic equation

$$2z^3 + pz^2 + qz + 16 = 0$$

where p and q are real, has roots α, β and γ.

It is given that $\alpha = 2 + 2\sqrt{3}\,\mathrm{i}$.

(a) (i) Write down another root, β, of the equation. *(1 mark)*

(ii) Find the third root, γ. *(3 marks)*

(iii) Find the values of p and q. *(3 marks)*

(b) (i) Express α in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. *(2 marks)*

(ii) Show that

$$(2 + 2\sqrt{3}\,\mathrm{i})^n = 4^n\left(\cos\frac{n\pi}{3} + \mathrm{i}\sin\frac{n\pi}{3}\right)$$ *(2 marks)*

(iii) Show that

$$\alpha^n + \beta^n + \gamma^n = 2^{2n+1}\cos\frac{n\pi}{3} + \left(-\frac{1}{2}\right)^n$$

where n is an integer. *(3 marks)*

4 A curve C is given parametrically by the equations

$$x = \frac{1}{2}\cosh 2t, \qquad y = 2\sinh t$$

(a) Express

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2$$

in terms of $\cosh t$. *(6 marks)*

(b) The arc of C from $t = 0$ to $t = 1$ is rotated through 2π radians about the x-axis.

(i) Show that S, the area of the curved surface generated, is given by

$$S = 8\pi\int_0^1 \sinh t\cosh^2 t\,\mathrm{d}t$$ *(2 marks)*

(ii) Find the exact value of S. *(2 marks)*

Turn over ▶

5 The sum to r terms, S_r, of a series is given by

$$S_r = r^2(r+1)(r+2)$$

Given that u_r is the rth term of the series whose sum is S_r, show that:

(a) (i) $u_1 = 6$; *(1 mark)*

(ii) $u_2 = 42$; *(1 mark)*

(iii) $u_n = n(n+1)(4n-1)$. *(3 marks)*

(b) Show that

$$\sum_{r=n+1}^{2n} u_r = 3n^2(n+1)(5n+2)$$ *(3 marks)*

6 (a) Show that the substitution $t = \tan\theta$ transforms the integral

$$\int \frac{d\theta}{9\cos^2\theta + \sin^2\theta}$$

into $$\int \frac{dt}{9+t^2}$$ *(3 marks)*

(b) Hence show that

$$\int_0^{\frac{\pi}{3}} \frac{d\theta}{9\cos^2\theta + \sin^2\theta} = \frac{\pi}{18}$$ *(3 marks)*

7 The sequence $u_1, u_2, u_3, \ldots$ is defined by

$$u_1 = 2, \quad u_{k+1} = 2u_k + 1$$

(a) Prove by induction that, for all $n \geqslant 1$,

$$u_n = 3 \times 2^{n-1} - 1$$ *(5 marks)*

(b) Show that

$$\sum_{r=1}^{n} u_r = u_{n+1} - (n+2)$$ *(3 marks)*

8 (a) (i) Show that $\omega = e^{\frac{2\pi i}{7}}$ is a root of the equation $z^7 = 1$. *(1 mark)*

(ii) Write down the five other non-real roots in terms of ω. *(2 marks)*

(b) Show that

$$1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 = 0$$ *(2 marks)*

(c) Show that:

(i) $\omega^2 + \omega^5 = 2\cos\frac{4\pi}{7}$; *(3 marks)*

(ii) $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}$. *(4 marks)*

END OF QUESTIONS

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Other Names										
Candidate Signature										

For Examiner's Use	
Examiner's Initials	
Question	Mark
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General Certificate of Education
Advanced Level Examination
June 2010

Mathematics MFP2

Unit Further Pure 2

Wednesday 9 June 2010 1.30 pm to 3.00 pm

For this paper you must have:
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed
- 1 hour 30 minutes

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- Write the question part reference (eg (a), (b)(i) etc) in the left-hand margin.
- You must answer the questions in the spaces provided. Do not write outside the box around each page.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work that you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 75.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

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Answer **all** questions in the spaces provided.

1 (a) Show that

$$9\sinh x - \cosh x = 4e^{x} - 5e^{-x}$$

(2 marks)

(b) Given that

$$9\sinh x - \cosh x = 8$$

find the exact value of $\tanh x$. *(7 marks)*

QUESTION PART REFERENCE

2 (a) Express $\dfrac{1}{r(r+2)}$ in partial fractions. *(3 marks)*

(b) Use the method of differences to find

$$\sum_{r=1}^{48} \frac{1}{r(r+2)}$$

giving your answer as a rational number. *(5 marks)*

QUESTION PART REFERENCE

Do not write outside the box

3 Two loci, L_1 and L_2, in an Argand diagram are given by

$$L_1 : |z + 1 + 3\mathrm{i}| = |z - 5 - 7\mathrm{i}|$$

$$L_2 : \arg z = \frac{\pi}{4}$$

(a) Verify that the point represented by the complex number $2 + 2\mathrm{i}$ is a point of intersection of L_1 and L_2. *(2 marks)*

(b) Sketch L_1 and L_2 on one Argand diagram. *(5 marks)*

(c) Shade on your Argand diagram the region satisfying

both $\qquad |z + 1 + 3\mathrm{i}| \leqslant |z - 5 - 7\mathrm{i}|$

and $\qquad \frac{\pi}{4} \leqslant \arg z \leqslant \frac{\pi}{2}$ *(2 marks)*

QUESTION PART REFERENCE

4 The roots of the cubic equation

$$z^3 - 2z^2 + pz + 10 = 0$$

are α, β and γ.

It is given that $\alpha^3 + \beta^3 + \gamma^3 = -4$.

(a) Write down the value of $\alpha + \beta + \gamma$. *(1 mark)*

(b) (i) Explain why $\alpha^3 - 2\alpha^2 + p\alpha + 10 = 0$. *(1 mark)*

(ii) Hence show that

$$\alpha^2 + \beta^2 + \gamma^2 = p + 13$$ *(4 marks)*

(iii) Deduce that $p = -3$. *(2 marks)*

(c) (i) Find the real root α of the cubic equation $z^3 - 2z^2 - 3z + 10 = 0$. *(2 marks)*

(ii) Find the values of β and γ. *(3 marks)*

QUESTION PART REFERENCE

5 (a) Using the identities

$$\cosh^2 t - \sinh^2 t = 1\,,\quad \tanh t = \frac{\sinh t}{\cosh t} \quad \text{and} \quad \operatorname{sech} t = \frac{1}{\cosh t}$$

show that:

(i) $\tanh^2 t + \operatorname{sech}^2 t = 1\,;$ *(2 marks)*

(ii) $\frac{\mathrm{d}}{\mathrm{d}t}(\tanh t) = \operatorname{sech}^2 t\,;$ *(3 marks)*

(iii) $\frac{\mathrm{d}}{\mathrm{d}t}(\operatorname{sech} t) = -\operatorname{sech} t \tanh t\,.$ *(3 marks)*

(b) A curve C is given parametrically by

$$x = \operatorname{sech} t\,,\quad y = 4 - \tanh t$$

(i) Show that the arc length, s, of C between the points where $t = 0$ and $t = \frac{1}{2}\ln 3$ is given by

$$s = \int_0^{\frac{1}{2}\ln 3} \operatorname{sech} t \,\mathrm{d}t$$ *(4 marks)*

(ii) Using the substitution $u = \mathrm{e}^t$, find the exact value of s. *(6 marks)*

QUESTION PART REFERENCE

6 (a) Show that $\dfrac{1}{(k+2)!} - \dfrac{k+1}{(k+3)!} = \dfrac{2}{(k+3)!}$. *(2 marks)*

(b) Prove by induction that, for all positive integers n,

$$\sum_{r=1}^{n} \frac{r \times 2^r}{(r+2)!} = 1 - \frac{2^{n+1}}{(n+2)!}$$

(6 marks)

QUESTION PART REFERENCE

7 (a) (i) Express each of the numbers $1+\sqrt{3}\,\mathrm{i}$ and $1-\mathrm{i}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$.
(3 marks)

(ii) Hence express

$$(1+\sqrt{3}\,\mathrm{i})^8(1-\mathrm{i})^5$$

in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$. *(3 marks)*

(b) Solve the equation

$$z^3=(1+\sqrt{3}\,\mathrm{i})^8(1-\mathrm{i})^5$$

giving your answers in the form $a\sqrt{2}\,\mathrm{e}^{\mathrm{i}\theta}$, where a is a positive integer and $-\pi<\theta\leqslant\pi$. *(4 marks)*

QUESTION PART REFERENCE

General Certificate of Education
Advanced Level Examination
January 2011

Mathematics MFP2

Unit Further Pure 2

Wednesday 19 January 2011 1.30 pm to 3.00 pm

For this paper you must have:
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed
- 1 hour 30 minutes

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- Write the question part reference (eg (a), (b)(i) etc) in the left-hand margin.
- You must answer the questions in the spaces provided. Do not write outside the box around each page.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work that you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 75.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

1 (a) Sketch on an Argand diagram the locus of points satisfying the equation

$$|z - 4 + 3\mathrm{i}| = 5$$ *(3 marks)*

(b) (i) Indicate on your diagram the point P representing z_1, where both

$$|z_1 - 4 + 3\mathrm{i}| = 5 \quad \text{and} \quad \arg z_1 = 0$$ *(1 mark)*

(ii) Find the value of $|z_1|$. *(1 mark)*

---

2 (a) Given that

$$u_r = \frac{1}{6}r(r+1)(4r+11)$$

show that

$$u_r - u_{r-1} = r(2r+3)$$ *(3 marks)*

(b) Hence find the sum of the first hundred terms of the series

$$1 \times 5 + 2 \times 7 + 3 \times 9 + \ldots + r(2r+3) + \ldots$$ *(3 marks)*

---

3 (a) Show that $(1+\mathrm{i})^3 = 2\mathrm{i} - 2$. *(2 marks)*

(b) The cubic equation

$$z^3 - (5+\mathrm{i})z^2 + (9+4\mathrm{i})z + k(1+\mathrm{i}) = 0$$

where k is a real constant, has roots α, β and γ.

It is given that $\alpha = 1 + \mathrm{i}$.

(i) Find the value of k. *(3 marks)*

(ii) Show that $\beta + \gamma = 4$. *(1 mark)*

(iii) Find the values of β and γ. *(5 marks)*

---

4 (a) Prove that the curve

$$y = 12\cosh x - 8\sinh x - x$$

has exactly one stationary point. *(7 marks)*

(b) Given that the coordinates of this stationary point are (a, b), show that $a + b = 9$. *(4 marks)*

---

5 (a) Given that $u = \sqrt{1 - x^2}$, find $\dfrac{\mathrm{d}u}{\mathrm{d}x}$. *(2 marks)*

(b) Use integration by parts to show that

$$\int_0^{\frac{\sqrt{3}}{2}} \sin^{-1} x \,\mathrm{d}x = a\sqrt{3}\,\pi + b$$

where a and b are rational numbers. *(6 marks)*

---

6 (a) Given that

$$x = \ln(\sec t + \tan t) - \sin t$$

show that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sin t \tan t$$ *(4 marks)*

(b) A curve is given parametrically by the equations

$$x = \ln(\sec t + \tan t) - \sin t, \quad y = \cos t$$

The length of the arc of the curve between the points where $t = 0$ and $t = \dfrac{\pi}{3}$ is denoted by s.

Show that $s = \ln p$, where p is an integer. *(6 marks)*

---

7 (a) Given that

$$f(k) = 12^k + 2 \times 5^{k-1}$$

show that

$$f(k+1) - 5f(k) = a \times 12^k$$

where a is an integer. *(3 marks)*

(b) Prove by induction that $12^n + 2 \times 5^{n-1}$ is divisible by 7 for all integers $n \geqslant 1$. *(4 marks)*

---

8 (a) Express in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$:

(i) $4(1 + i\sqrt{3})$;

(ii) $4(1 - i\sqrt{3})$. *(3 marks)*

(b) The complex number z satisfies the equation

$$(z^3 - 4)^2 = -48$$

Show that $z^3 = 4 \pm 4\sqrt{3}\,i$. *(2 marks)*

(c) (i) Solve the equation

$$(z^3 - 4)^2 = -48$$

giving your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. *(5 marks)*

(ii) Illustrate the roots on an Argand diagram. *(3 marks)*

(d) (i) Explain why the sum of the roots of the equation

$$(z^3 - 4)^2 = -48$$

is zero. *(1 mark)*

(ii) Deduce that $\cos\frac{\pi}{9} + \cos\frac{3\pi}{9} + \cos\frac{5\pi}{9} + \cos\frac{7\pi}{9} = \frac{1}{2}$. *(3 marks)*

General Certificate of Education
Advanced Level Examination
June 2011

Mathematics MFP2

Unit Further Pure 2

Monday 13 June 2011 9.00 am to 10.30 am

For this paper you must have:
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed
- 1 hour 30 minutes

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- Write the question part reference (eg (a), (b)(i) etc) in the left-hand margin.
- You must answer the questions in the spaces provided. Do not write outside the box around each page.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work that you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 75.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

1 (a) Draw on the same Argand diagram:

(i) the locus of points for which

$$|z - 2 - 5\mathrm{i}| = 5$$ *(3 marks)*

(ii) the locus of points for which

$$\arg(z + 2\mathrm{i}) = \frac{\pi}{4}$$ *(3 marks)*

(b) Indicate on your diagram the set of points satisfying both

$$|z - 2 - 5\mathrm{i}| \leqslant 5$$

and

$$\arg(z + 2\mathrm{i}) = \frac{\pi}{4}$$ *(2 marks)*

---

2 (a) Use the definitions of $\cosh\theta$ and $\sinh\theta$ in terms of e^{θ} to show that

$$\cosh x \cosh y - \sinh x \sinh y = \cosh(x - y)$$ *(4 marks)*

(b) It is given that x satisfies the equation

$$\cosh(x - \ln 2) = \sinh x$$

(i) Show that $\tanh x = \frac{5}{7}$. *(4 marks)*

(ii) Express x in the form $\frac{1}{2}\ln a$. *(2 marks)*

---

3 (a) Show that

$$(r + 1)! - (r - 1)! = (r^2 + r - 1)(r - 1)!$$ *(2 marks)*

(b) Hence show that

$$\sum_{r=1}^{n}(r^2 + r - 1)(r - 1)! = (n + 2)n! - 2$$ *(4 marks)*

4 The cubic equation

$$z^3 - 2z^2 + k = 0 \qquad (k \neq 0)$$

has roots α, β and γ.

(a) (i) Write down the values of $\alpha + \beta + \gamma$ and $\alpha\beta + \beta\gamma + \gamma\alpha$. *(2 marks)*

(ii) Show that $\alpha^2 + \beta^2 + \gamma^2 = 4$. *(2 marks)*

(iii) Explain why $\alpha^3 - 2\alpha^2 + k = 0$. *(1 mark)*

(iv) Show that $\alpha^3 + \beta^3 + \gamma^3 = 8 - 3k$. *(2 marks)*

(b) Given that $\alpha^4 + \beta^4 + \gamma^4 = 0$:

(i) show that $k = 2$; *(4 marks)*

(ii) find the value of $\alpha^5 + \beta^5 + \gamma^5$. *(3 marks)*

---

5 (a) The arc of the curve $y^2 = x^2 + 8$ between the points where $x = 0$ and $x = 6$ is rotated through 2π radians about the x-axis. Show that the area S of the curved surface formed is given by

$$S = 2\sqrt{2}\pi \int_0^6 \sqrt{x^2 + 4}\,\mathrm{d}x$$ *(5 marks)*

(b) By means of the substitution $x = 2\sinh\theta$, show that

$$S = \pi(24\sqrt{5} + 4\sqrt{2}\sinh^{-1} 3)$$ *(8 marks)*

---

6 (a) Show that

$$(k+1)(4(k+1)^2 - 1) = 4k^3 + 12k^2 + 11k + 3$$ *(2 marks)*

(b) Prove by induction that, for all integers $n \geqslant 1$,

$$1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{1}{3}n(4n^2 - 1)$$ *(6 marks)*

Turn over ▶

7 (a) (i) Use de Moivre's Theorem to show that

$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$

and find a similar expression for $\sin 5\theta$. *(5 marks)*

(ii) Deduce that

$$\tan 5\theta = \frac{\tan\theta(5 - 10\tan^2\theta + \tan^4\theta)}{1 - 10\tan^2\theta + 5\tan^4\theta}$$ *(3 marks)*

(b) Explain why $t = \tan\dfrac{\pi}{5}$ is a root of the equation

$$t^4 - 10t^2 + 5 = 0$$

and write down the three other roots of this equation in trigonometrical form. *(3 marks)*

(c) Deduce that

$$\tan\frac{\pi}{5}\tan\frac{2\pi}{5} = \sqrt{5}$$ *(5 marks)*

END OF QUESTIONS

General Certificate of Education
Advanced Level Examination
January 2012

Mathematics MFP2

Unit Further Pure 2

Friday 20 January 2012 1.30 pm to 3.00 pm

For this paper you must have:
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed
- 1 hour 30 minutes

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- Write the question part reference (eg (a), (b)(i) etc) in the left-hand margin.
- You must answer the questions in the spaces provided. Do not write outside the box around each page.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work that you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 75.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.
- You do not necessarily need to use all the space provided.

1 (a) Show, by means of a sketch, that the curves with equations

$$y = \sinh x$$

and $$y = \operatorname{sech} x$$

have exactly one point of intersection. *(4 marks)*

(b) Find the x-coordinate of this point of intersection, giving your answer in the form $a \ln b$. *(4 marks)*

---

2 (a) Draw on an Argand diagram the locus L of points satisfying the equation $\arg z = \dfrac{\pi}{6}$. *(1 mark)*

(b) (i) A circle C, of radius 6, has its centre lying on L and touches the line $\operatorname{Re}(z) = 0$. Draw C on your Argand diagram from part **(a)**. *(2 marks)*

(ii) Find the equation of C, giving your answer in the form $|z - z_0| = k$. *(3 marks)*

(iii) The complex number z_1 lies on C and is such that $\arg z_1$ has its least possible value. Find $\arg z_1$, giving your answer in the form $p\pi$, where $-1 < p \leqslant 1$. *(2 marks)*

---

3 A curve has cartesian equation

$$y = \tfrac{1}{2}\ln(\tanh x)$$

(a) Show that

$$\frac{dy}{dx} = \frac{1}{\sinh 2x}$$ *(4 marks)*

(b) The points A and B on the curve have x-coordinates $\ln 2$ and $\ln 4$ respectively. Find the arc length AB, giving your answer in the form $p \ln q$, where p and q are rational numbers. *(8 marks)*

4 The sequence $u_1, u_2, u_3, \ldots$ is defined by

$$u_1 = \frac{3}{4} \qquad u_{n+1} = \frac{3}{4 - u_n}$$

Prove by induction that, for all $n \geqslant 1$,

$$u_n = \frac{3^{n+1} - 3}{3^{n+1} - 1}$$

(6 marks)

---

5 Find the smallest positive integer values of p and q for which

$$\frac{\left(\cos\frac{\pi}{8} + \mathrm{i}\sin\frac{\pi}{8}\right)^p}{\left(\cos\frac{\pi}{12} - \mathrm{i}\sin\frac{\pi}{12}\right)^q} = \mathrm{i}$$

(7 marks)

---

6 (a) Express $7 + 4x - 2x^2$ in the form $a - b(x - c)^2$, where a, b and c are integers.

(2 marks)

(b) By means of a suitable substitution, or otherwise, find the exact value of

$$\int_1^{\frac{5}{2}} \frac{\mathrm{d}x}{\sqrt{7 + 4x - 2x^2}}$$

(6 marks)

Turn over ▶

7 The numbers α, β and γ satisfy the equations

$$\alpha^2 + \beta^2 + \gamma^2 = -10 - 12\mathrm{i}$$
$$\alpha\beta + \beta\gamma + \gamma\alpha = 5 + 6\mathrm{i}$$

(a) Show that $\alpha + \beta + \gamma = 0$. *(2 marks)*

(b) The numbers α, β and γ are also the roots of the equation

$$z^3 + pz^2 + qz + r = 0$$

Write down the value of p and the value of q. *(2 marks)*

(c) It is also given that $\alpha = 3\mathrm{i}$.

(i) Find the value of r. *(3 marks)*

(ii) Show that β and γ are the roots of the equation

$$z^2 + 3\mathrm{i}z - 4 + 6\mathrm{i} = 0$$ *(2 marks)*

(iii) Given that β is real, find the values of β and γ. *(3 marks)*

---

8 (a) Write down the five roots of the equation $z^5 = 1$, giving your answers in the form $\mathrm{e}^{\mathrm{i}\theta}$, where $-\pi < \theta \leqslant \pi$. *(1 mark)*

(b) Hence find the four linear factors of

$$z^4 + z^3 + z^2 + z + 1$$ *(3 marks)*

(c) Deduce that

$$z^2 + z + 1 + z^{-1} + z^{-2} = \left(z - 2\cos\frac{2\pi}{5} + z^{-1}\right)\left(z - 2\cos\frac{4\pi}{5} + z^{-1}\right)$$ *(4 marks)*

(d) Use the substitution $z + z^{-1} = w$ to show that $\cos\frac{2\pi}{5} = \frac{\sqrt{5} - 1}{4}$. *(6 marks)*

General Certificate of Education
Advanced Level Examination
June 2012

Mathematics MFP2

Unit Further Pure 2

Thursday 31 May 2012 9.00 am to 10.30 am

For this paper you must have:
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed
- 1 hour 30 minutes

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- Write the question part reference (eg (a), (b)(i) etc) in the left-hand margin.
- You must answer each question in the space provided for that question. If you require extra space, use an AQA supplementary answer book; do **not** use the space provided for a different question.
- Do not write outside the box around each page.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work that you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 75.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.
- You do not necessarily need to use all the space provided.

1 (a) Sketch the curve $y = \cosh x$. *(1 mark)*

(b) Solve the equation

$$6\cosh^2 x - 7\cosh x - 5 = 0$$

giving your answers in logarithmic form. *(6 marks)*

2 (a) Draw on the Argand diagram below:

(i) the locus of points for which

$$|z - 2 - 3\mathrm{i}| = 2$$ *(3 marks)*

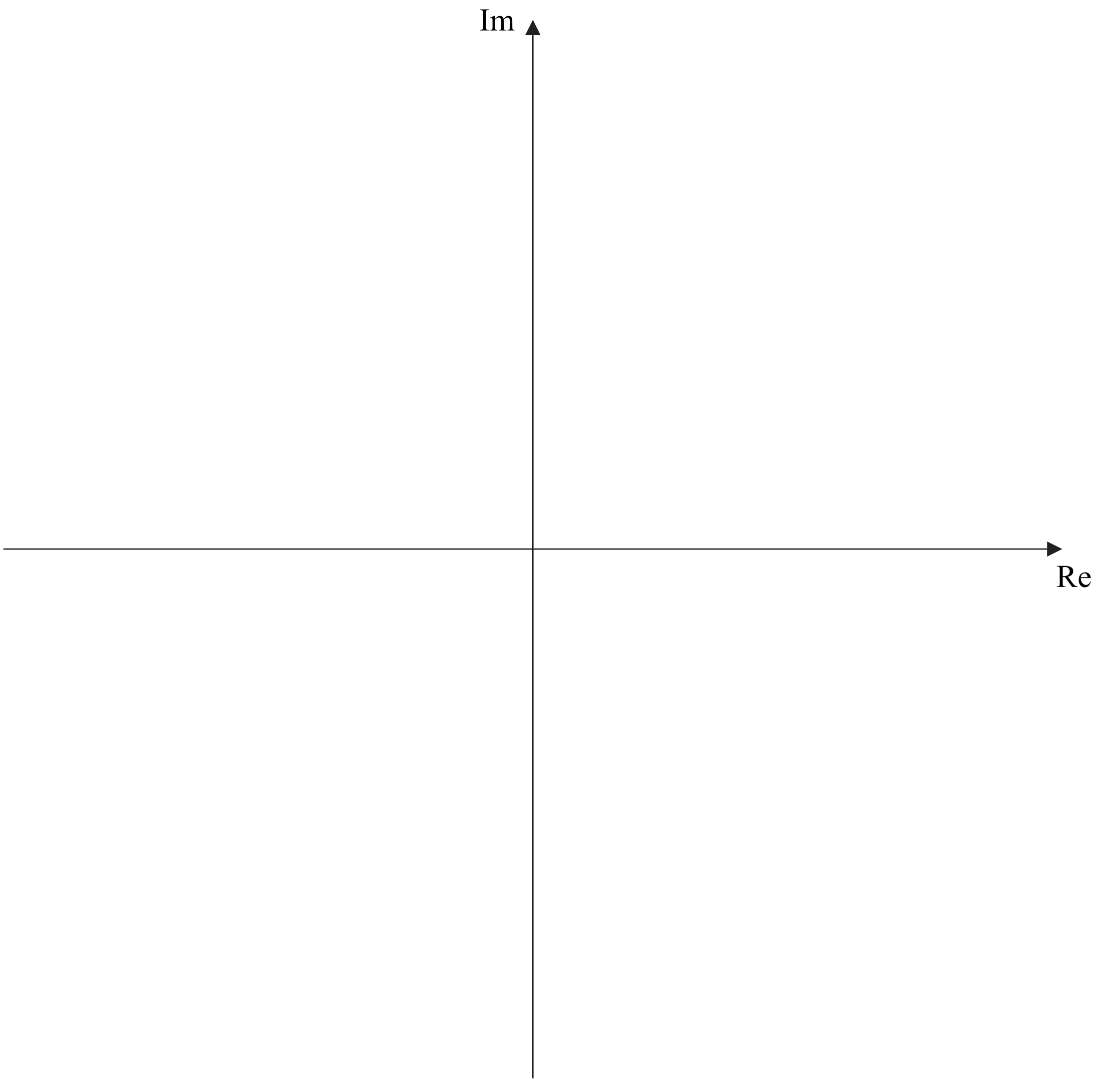
(ii) the locus of points for which

$$|z + 2 - \mathrm{i}| = |z - 2|$$ *(3 marks)*

(b) Indicate on your diagram the points satisfying both

$$|z - 2 - 3\mathrm{i}| = 2$$

and $$|z + 2 - \mathrm{i}| \leqslant |z - 2|$$ *(1 mark)*



Turn over ▶

3 (a) Show that

$$\frac{2^{r+1}}{r+2} - \frac{2^r}{r+1} = \frac{r2^r}{(r+1)(r+2)}$$ *(3 marks)*

(b) Hence find

$$\sum_{r=1}^{30} \frac{r2^r}{(r+1)(r+2)}$$

giving your answer in the form $2^n - 1$, where n is an integer. *(3 marks)*

---

4 The cubic equation

$$z^3 + pz + q = 0$$

has roots α, β and γ.

(a) (i) Write down the value of $\alpha + \beta + \gamma$. *(1 mark)*

(ii) Express $\alpha\beta\gamma$ in terms of q. *(1 mark)*

(b) Show that

$$\alpha^3 + \beta^3 + \gamma^3 = 3\alpha\beta\gamma$$ *(3 marks)*

(c) Given that $\alpha = 4 + 7\mathrm{i}$ and that p and q are real, find the values of:

(i) β and γ; *(2 marks)*

(ii) p and q. *(3 marks)*

(d) Find a cubic equation with integer coefficients which has roots $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$. *(3 marks)*

---

5 The function f, where $\mathrm{f}(x) = \sec x$, has domain $0 \leqslant x < \frac{\pi}{2}$ and has inverse function f^{-1}, where $\mathrm{f}^{-1}(x) = \sec^{-1} x$.

(a) Show that

$$\sec^{-1} x = \cos^{-1} \frac{1}{x}$$ *(2 marks)*

(b) Hence show that

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sec^{-1} x) = \frac{1}{\sqrt{x^4 - x^2}}$$ *(4 marks)*

6 (a) Show that

$$\frac{1}{4}(\cosh 4x + 2\cosh 2x + 1) = \cosh^2 x \cosh 2x$$ *(3 marks)*

(b) Show that, if $y = \cosh^2 x$, then

$$1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = \cosh^2 2x$$ *(3 marks)*

(c) The arc of the curve $y = \cosh^2 x$ between the points where $x = 0$ and $x = \ln 2$ is rotated through 2π radians about the x-axis. Show that the area S of the curved surface formed is given by

$$S = \frac{\pi}{256}(a \ln 2 + b)$$

where a and b are integers. *(7 marks)*

---

7 (a) Prove by induction that, for all integers $n \geqslant 1$,

$$\frac{3}{1^2 \times 2^2} + \frac{5}{2^2 \times 3^2} + \frac{7}{3^2 \times 4^2} + \dots + \frac{2n+1}{n^2(n+1)^2} = 1 - \frac{1}{(n+1)^2}$$ *(7 marks)*

(b) Find the smallest integer n for which the sum of the series differs from 1 by less than 10^{-5}. *(2 marks)*

Turn over ▶

8 (a) Use De Moivre's Theorem to show that, if $z = \cos\theta + \mathrm{i}\sin\theta$, then

$$z^n + \frac{1}{z^n} = 2\cos n\theta$$ *(3 marks)*

(b) (i) Expand $\left(z^2 + \frac{1}{z^2}\right)^4$. *(1 mark)*

(ii) Show that

$$\cos^4 2\theta = A\cos 8\theta + B\cos 4\theta + C$$

where A, B and C are rational numbers. *(4 marks)*

(c) Hence solve the equation

$$8\cos^4 2\theta = \cos 8\theta + 5$$

for $0 \leqslant \theta \leqslant \pi$, giving each solution in the form $k\pi$. *(3 marks)*

(d) Show that

$$\int_0^{\frac{\pi}{2}} \cos^4 2\theta \,\mathrm{d}\theta = \frac{3\pi}{16}$$ *(3 marks)*

General Certificate of Education
Advanced Level Examination
January 2013

Mathematics MFP2

Unit Further Pure 2

Wednesday 23 January 2013 9.00 am to 10.30 am

For this paper you must have:
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed
- 1 hour 30 minutes

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- Write the question part reference (eg (a), (b)(i) etc) in the left-hand margin.
- You must answer each question in the space provided for that question. If you require extra space, use an AQA supplementary answer book; do **not** use the space provided for a different question.
- Do not write outside the box around each page.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work that you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 75.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.
- You do not necessarily need to use all the space provided.

1 (a) Show that

$$12\cosh x - 4\sinh x = 4\mathrm{e}^{x} + 8\mathrm{e}^{-x}$$ *(2 marks)*

(b) Solve the equation

$$12\cosh x - 4\sinh x = 33$$

giving your answers in the form $k\ln 2$. *(5 marks)*

---

2 Two loci, L_1 and L_2, in an Argand diagram are given by

$$L_1 : |z + 6 - 5\mathrm{i}| = 4\sqrt{2}$$

$$L_2 : \quad \arg(z + \mathrm{i}) = \frac{3\pi}{4}$$

The point P represents the complex number $-2 + \mathrm{i}$.

(a) Verify that the point P is a point of intersection of L_1 and L_2. *(2 marks)*

(b) Sketch L_1 and L_2 on one Argand diagram. *(6 marks)*

(c) The point Q is also a point of intersection of L_1 and L_2. Find the complex number that is represented by Q. *(2 marks)*

---

3 (a) Show that $\dfrac{1}{5r-2} - \dfrac{1}{5r+3} = \dfrac{A}{(5r-2)(5r+3)}$, stating the value of the constant A. *(2 marks)*

(b) Hence use the method of differences to show that

$$\sum_{r=1}^{n} \frac{1}{(5r-2)(5r+3)} = \frac{n}{3(5n+3)}$$ *(4 marks)*

(c) Find the value of

$$\sum_{r=1}^{\infty} \frac{1}{(5r-2)(5r+3)}$$ *(1 mark)*

4 The roots of the equation

$$z^3 - 5z^2 + kz - 4 = 0$$

are α, β and γ.

(a) (i) Write down the value of $\alpha + \beta + \gamma$ and the value of $\alpha\beta\gamma$. *(2 marks)*

(ii) Hence find the value of $\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2$. *(2 marks)*

(b) The value of $\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2$ is -4.

(i) Explain why α, β and γ cannot all be real. *(1 mark)*

(ii) By considering $(\alpha\beta + \beta\gamma + \gamma\alpha)^2$, find the possible values of k. *(4 marks)*

---

5 (a) Using the definition $\tanh y = \dfrac{e^y - e^{-y}}{e^y + e^{-y}}$, show that, for $|x| < 1$,

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$ *(3 marks)*

(b) Hence, or otherwise, show that $\dfrac{d}{dx}(\tanh^{-1} x) = \dfrac{1}{1-x^2}$. *(3 marks)*

(c) Use integration by parts to show that

$$\int_0^{\frac{1}{2}} 4\tanh^{-1} x \, dx = \ln\left(\frac{3^m}{2^n}\right)$$

where m and n are positive integers. *(5 marks)*

---

6 A curve is defined parametrically by

$$x = t^3 + 5, \qquad y = 6t^2 - 1$$

The arc length between the points where $t = 0$ and $t = 3$ on the curve is s.

(a) Show that $s = \displaystyle\int_0^3 3t\sqrt{t^2 + A}\, dt$, stating the value of the constant A. *(4 marks)*

(b) Hence show that $s = 61$. *(4 marks)*

Turn over ▶

7 The polynomial $\mathrm{p}(n)$ is given by $\mathrm{p}(n) = (n-1)^3 + n^3 + (n+1)^3$.

(a) (i) Show that $\mathrm{p}(k+1) - \mathrm{p}(k)$, where k is a positive integer, is a multiple of 9. *(3 marks)*

(ii) Prove by induction that $\mathrm{p}(n)$ is a multiple of 9 for all integers $n \geqslant 1$. *(4 marks)*

(b) Using the result from part **(a)(ii)**, show that $n(n^2+2)$ is a multiple of 3 for any positive integer n. *(2 marks)*

---

8 (a) Express $-4 + 4\sqrt{3}\,\mathrm{i}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. *(3 marks)*

(b) (i) Solve the equation $z^3 = -4 + 4\sqrt{3}\,\mathrm{i}$, giving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. *(4 marks)*

(ii) The roots of the equation $z^3 = -4 + 4\sqrt{3}\,\mathrm{i}$ are represented by the points P, Q and R on an Argand diagram.

Find the area of the triangle PQR, giving your answer in the form $k\sqrt{3}$, where k is an integer. *(3 marks)*

(c) By considering the roots of the equation $z^3 = -4 + 4\sqrt{3}\,\mathrm{i}$, show that

$$\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} + \cos\frac{8\pi}{9} = 0$$

(4 marks)

General Certificate of Education
Advanced Level Examination
June 2013

Mathematics MFP2

Unit Further Pure 2

Thursday 6 June 2013 9.00 am to 10.30 am

For this paper you must have:
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed
- 1 hour 30 minutes

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
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- Do not write outside the box around each page.
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- Do all rough work in this book. Cross through any work that you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 75.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.
- You do not necessarily need to use all the space provided.

Do not write outside the box

1 (a) Sketch on an Argand diagram the locus of points satisfying the equation

$$|z - 6\mathrm{i}| = 3$$ *(3 marks)*

(b) It is given that z satisfies the equation $|z - 6\mathrm{i}| = 3$.

(i) Write down the greatest possible value of $|z|$. *(1 mark)*

(ii) Find the greatest possible value of $\arg z$, giving your answer in the form $p\pi$, where $-1 < p \leqslant 1$. *(3 marks)*

---

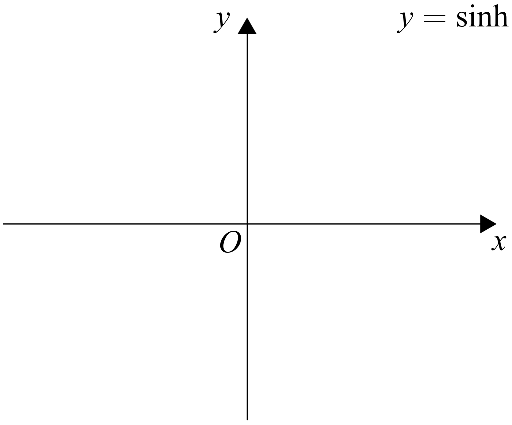
2 (a) (i) Sketch on the axes below the graphs of $y = \sinh x$ and $y = \cosh x$. *(3 marks)*

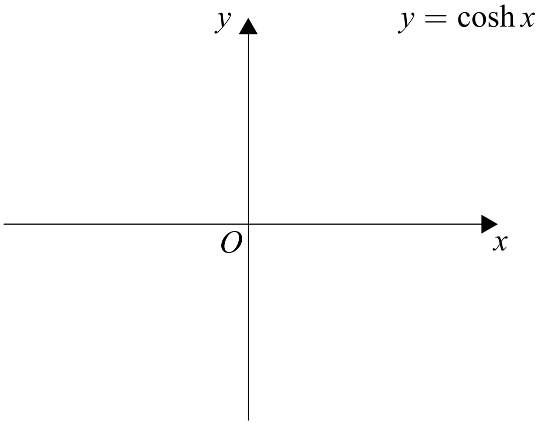
(ii) Use your graphs to explain why the equation

$$(k + \sinh x)\cosh x = 0$$

where k is a constant, has exactly one solution. *(1 mark)*

(b) A curve C has equation $y = 6\sinh x + \cosh^2 x$. Show that C has only one stationary point and show that its y-coordinate is an integer. *(5 marks)*





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3 The sequence $u_1, u_2, u_3, \ldots$ is defined by

$$u_1 = 2, \qquad u_{n+1} = \frac{5u_n - 3}{3u_n - 1}$$

Prove by induction that, for all integers $n \geqslant 1$,

$$u_n = \frac{3n + 1}{3n - 1}$$ *(6 marks)*

4 (a) Given that $f(r) = r^2(2r^2 - 1)$, show that

$$f(r) - f(r-1) = (2r-1)^3$$

(3 marks)

(b) Use the method of differences to show that

$$\sum_{r=n+1}^{2n} (2r-1)^3 = 3n^2(10n^2 - 1)$$

(4 marks)

---

5 The cubic equation

$$z^3 + pz^2 + qz + 37 - 36\mathrm{i} = 0$$

where p and q are constants, has three complex roots, α, β and γ.

It is given that $\beta = -2 + 3\mathrm{i}$ and $\gamma = 1 + 2\mathrm{i}$.

(a) (i) Write down the value of $\alpha\beta\gamma$. *(1 mark)*

(ii) Hence show that $(8 + \mathrm{i})\alpha = 37 - 36\mathrm{i}$. *(2 marks)*

(iii) Hence find α, giving your answer in the form $m + n\mathrm{i}$, where m and n are integers. *(3 marks)*

(b) Find the value of p. *(1 mark)*

(c) Find the value of the complex number q. *(2 marks)*

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6 (a) Show that $\dfrac{1}{5\cosh x - 3\sinh x} = \dfrac{\mathrm{e}^x}{m + \mathrm{e}^{2x}}$, where m is an integer. *(3 marks)*

(b) Use the substitution $u = \mathrm{e}^x$ to show that

$$\int_0^{\ln 2} \frac{1}{5\cosh x - 3\sinh x}\,\mathrm{d}x = \frac{\pi}{8} - \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right)$$

(5 marks)

Turn over ▶

7 (a) (i) Show that

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(2u\sqrt{1+4u^2}+\sinh^{-1}2u\right)=k\sqrt{1+4u^2}$$

where k is an integer. *(4 marks)*

(ii) Hence show that

$$\int_0^1 \sqrt{1+4u^2}\,\mathrm{d}u = p\sqrt{5}+q\sinh^{-1}2$$

where p and q are rational numbers. *(2 marks)*

(b) The arc of the curve with equation $y=\frac{1}{2}\cos 4x$ between the points where $x=0$ and $x=\frac{\pi}{8}$ is rotated through 2π radians about the x-axis.

(i) Show that the area S of the curved surface formed is given by

$$S=\pi\int_0^{\frac{\pi}{8}}\cos 4x\sqrt{1+4\sin^2 4x}\,\mathrm{d}x$$ *(2 marks)*

(ii) Use the substitution $u=\sin 4x$ to find the exact value of S. *(4 marks)*

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8 (a) (i) Use de Moivre's theorem to show that

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta+\sin^4\theta$$

and find a similar expression for $\sin 4\theta$. *(5 marks)*

(ii) Deduce that

$$\tan 4\theta=\frac{4\tan\theta-4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta}$$ *(3 marks)*

(b) Explain why $t=\tan\frac{\pi}{16}$ is a root of the equation

$$t^4+4t^3-6t^2-4t+1=0$$

and write down the three other roots in trigonometric form. *(4 marks)*

(c) Hence show that

$$\tan^2\frac{\pi}{16}+\tan^2\frac{3\pi}{16}+\tan^2\frac{5\pi}{16}+\tan^2\frac{7\pi}{16}=28$$ *(5 marks)*